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ON A FORMULA OF INTERPOLATION.

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(Presented at the 1900 December Meeting of the American Mathematical Society.)

§1. INTRODUCTORY STATEMENTS.

Last July the writer obtained an extension of the formula $f(x+n)=(1+\triangle)^n f(x)$ for a positive integral n, of finite differences, to any real value of n and within the limits of convergence of the series, but only for the simplest value of the operator $(1+\triangle)^{\nu/\lambda}$. The other roots of the operator would be found by multiplying the simplest one by the other ν/λth roots of unity. Neither did the writer contemplate an extension to imaginary values of n. That would have to be done by definition. He has recently corresponded with Dr. McClintock and Prof. H. L. Rice, both of whom kindly communicated the fundamentals of their views.

Dr. McClintock has treated the question in this manner: (*American Journal of Mathematics*, 1879, Vol. II, page 117) "I find it better to define E^h as a compound symbol representing that simple operation which changes $(\phi)x$ into $\phi(x+h)$, whatever be the meaning of h. In this light we must regard E, when without an index, as an abbreviated form of E^1." The reason which he gave for his definition was as follows: "Yet if $E\phi(x)=\phi(x+1)$ expresses the definition of E, it will require considerable labor to prove that in all cases $E^h\phi(x)=\phi(x+h)$, and then only when h expresses some positive or negative quantity; and the argument will not be free from ambiguity, since, for example, it might be hard to prove that $E^{\frac{1}{2}}\,\phi(x)$ cannot be its own opposite, namely, $-\phi(x+\frac{1}{2})$."

Professor Rice writes as follows: "In my work on 'Interpolation' I have proved the truth of the fundamental formula

$$f(x+y)=(1+\triangle)^y f(x)$$

in so far as it *can* be proven. I have put the matter in this form:

(1). I have rigorously shown that the formula is true for *any* value of y *provided* the *differences* become ***absolutely constant*** at some particular order.

(2). I have further proved that if the differences become *sensibly*, but *not* absolutely constant at some order not too remote, then the formula is applicable, and will give as close an approximation to the truth as the nature and accuracy of the tabular quantities may permit."

His proof is obtained through the medium of Taylor's formula, two different series in terms of finite differences being related to the same Taylor's series, and thus to each other. On the other hand the proof to be given here is rather by the immediate and direct relation of two series in terms of finite differences to each other, and independently of any Taylor's series. It is first proved that the formula $f(x+y)=(1+\triangle)^y f(x)$ holds for a positive commensurable fraction y, then for a negative integer, next for a negative commensurable fraction, and lastly for an incommensurable y.

Whatever the value of m, the following abbreviation is used throughout this paper:

$$\begin{bmatrix} m \\ r \end{bmatrix}=\frac{m(m-1)\ldots\ldots(m-r+1)}{r!} \quad\ldots\ldots\ldots\ldots\ldots\ (1).$$

§2. THE TWO SERIES OF THE SAME FUNCTION, AND THEIR DIFFERENCES.

Two series of the same function are taken:

$$f(x+0)=u_0,\ f(x+1)=u_1,\ \ldots\ldots\ f(x+r)=u_r,\ \ldots\ldots \text{ with differences } \triangle n\text{'s}\ldots\ldots(2).$$

$$f(x+0)=v_0.\ f(x+\frac{1}{\lambda})=v_1,\ \ldots\ldots f(x+\frac{\nu}{\lambda})=v_\nu,\ \ldots\ldots \text{ with differences } \delta v\text{'s}\ldots\ldots(3).$$

Here ν and λ are both integers. Analogously to $u_r=(1+\triangle)^r u_0$, we can also prove,

$$v_\nu=(1+\delta)^\nu\, v_0\ldots\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

since we may write

$$v_0=f(x,\ 0),\ v_1=f(x,\ 1),\ \ldots\ldots\ v_\nu=f(x,\ \nu).$$

Also,

$$v_0=v_1,\ v_\lambda=u_1,\ \ldots\ldots\ v_{r\lambda}=u_r,\ \ldots\ldots,$$

and for example,

$$v_1 - v_0 + v_2 - v_1 + \dots . v_\lambda - v_{\lambda-1} = v_\lambda - v_0 = u_1 - u_0$$
$$= \delta v_0 + \delta v_1 + \dots . \delta_{\lambda-1} = \triangle u_0 \dots\dots\dots\dots (5).$$

and similarly for the other $\varDelta u$'s. As the latter are the sum of λ terms of the δv's, we have by the formula for such a sum,

$$\varDelta u_0 = \binom{\lambda}{1} \delta v_0 + \binom{\lambda}{2} \delta^2 v_0 + \dots . \binom{\lambda}{\lambda} \delta^\lambda v_0.$$

$$\varDelta u_1 = \binom{\lambda}{1} \delta v_\lambda + \binom{\lambda}{2} \delta^2 v_\lambda + \dots . \binom{\lambda}{\lambda} \delta^\lambda v_\lambda .$$

. . . .

$$\varDelta u_r = \binom{\lambda}{1} \delta v_{r\lambda} + \binom{\lambda}{2} \delta^2 v_{r\lambda} + \dots . \binom{\lambda}{\lambda} \delta^\lambda v_{r\lambda}.$$

. . . . (6).

$$\varDelta^2 u_0 = \binom{\lambda}{1} (\delta v_\sigma - \delta v_0) + \binom{\lambda}{2} (\delta^2 v_\lambda - \delta^2 v_0) + \dots \binom{\lambda}{\lambda} (\delta^\lambda v_\lambda - \delta^\lambda v_0).$$

. . . .

$$\varDelta^{r+1} u_0 = \binom{\lambda}{1} (\delta v_\lambda - \delta v_0)_r + \binom{\lambda}{2} (\delta^2 v_\lambda - \delta^2 v_0)_r + \dots . \binom{\lambda}{\lambda} (\delta^\lambda v_\lambda - \delta^\lambda v_0)_r$$

where $(\delta^\kappa v_\lambda - \delta^\kappa v_0)_r = \delta^\kappa v_{r\lambda} - \binom{r}{1} \delta^\kappa v_{(r-1)\lambda} + \binom{r}{2} \delta^\kappa v_{(r-2)\lambda} - \dots .$

Expressing $v_{r\lambda}$, $v_{(r-1)\lambda}$, ... etc., in terms of v_0,

$$\varDelta^{r+1} u_0 = \binom{\lambda}{1} (\varDelta^r(0, r) \delta^{r+1} v_0 + \varDelta^r(0, r+1) \delta^{r+2} v_0 + \dots .)$$
$$+ \binom{\lambda}{2} (\varDelta^r(0, r) \delta^{r+2} v_0 + \dots) + \dots . (7)$$

where

$$(x, H) = \binom{x\lambda}{H}, \quad \varDelta^r(0, H) = \binom{r\lambda}{H} - \binom{r}{1} \binom{(r-1)\lambda}{H} + \dots . (-1)^{r-1} \binom{r}{r-1} \binom{\lambda}{H},$$

and all the coefficients of the δv's in (7) of lower order than the $(r+1)$st are zero by means of the relation

$$\varDelta^r(x-r, H) = 0, \qquad (8)$$

if $r > H$. This is seen by considering the fact that

$$\left[\begin{matrix}(x-r)\lambda\\ H\end{matrix}\right]=\frac{(x-r)\lambda(x-r)\lambda-1)\dots((x-r)\lambda-H+1)}{H\,!}$$

is of degree H in x; therefore $\Delta^r(x-r,\ H)$ is of degree $H-r$ in x, and therefore zero, if $r>H$. We may also notice that,

$$\Delta^r(0,\ r)=A_r\lambda^r=\lambda^r,$$

$$\Delta^r(0,\ r+1)=A_{r+1}\lambda^r(\lambda-1)=\frac{r}{2}\lambda^r(\lambda-1), \tag{9}$$

where A_r, and A_{r+1} are the coefficients of x^r and x^{r+1} in $(e^x-1)^r$.* From this

$$\Delta^{r+1}u_0=\lambda^{r+1}\delta^{r+1}v_0+\frac{r}{2}\lambda^{r+1}(\lambda-1)\delta^{r+2}v_0+\dots \tag{10}$$

that is, each $\Delta^\kappa u_0$ is expressed in terms of $\lambda^\kappa\,\delta^\kappa\,v_0$, and higher differences than those of order κ in the δv's. We see:

If the κth order of differences in the δv's is constant, $\delta^{\kappa+r}v_0=0$. and $\Delta^\kappa u_0=\lambda^\kappa\,\delta^\kappa\,v_0$ exactly and $\Delta^{\kappa+r}u_0=0$, and conversely if $\Delta^{\kappa+r}u_0=0$, it follows that $\delta^{\kappa+r}v_0=0$. and $\Delta^\kappa\,u_0=\lambda^\kappa\,\delta^\kappa\,v_0$, for a series of linear functions of the δv's beginning with the $(\kappa+1)$st order of differences will be zero independently of λ. Hence the order of constancy is the same for both series.

§3. THE SERIES $(1+\Delta)^{\nu/\lambda}f(x)$.

The series, $(\nu^1=\frac{\nu}{\lambda})$,

$$u_0+\left[\begin{matrix}\nu^1\\ 1\end{matrix}\right]\Delta u_0+\left[\begin{matrix}\nu^1\\ 2\end{matrix}\right]\Delta^2u_0+\dots\left[\begin{matrix}\nu^1\\ r\end{matrix}\right]\Delta^ru_0+\dots \tag{11}$$

within the limits of convergence, is taken as starting point in the proof. By convention, this development may be neatly expressed symbolically as $(1+\Delta)^{\nu/\lambda}u_0$, if it is agreed to understand that only the simplest expansion of the operator is taken, and not any one of the values it might have from its similarity to a fractional power of a real binomial $1+x$. This however will be mere notation and does not form any necessary part of the proof.

§4. THE CONVERGENCY OF THE SERIES.

There are various tests of convergency of series. One of the most elementary gives the following result: If beginning with and after a value r,

$$\operatorname{mod}\frac{\nu^1-r+1}{r}\,\frac{\Delta^ru_0}{\Delta^{r-1}u_0}<\kappa<1, \tag{12}$$

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*See "Note on Integral and Integro-Geometric Series" by the author, *Annals of Mathematics*, October, 1897, page 184.

the series is convergent. The series is certainly convergent at least:

1. If the $\varkappa$th order of differences of the Δu's is constant; then $\Delta^{\kappa+1}u_0 = \Delta^{\kappa+2}u_0 = \ldots\ldots = 0$, and the series breaks off and is a finite series.
2. If

$$2 > \operatorname{Lim}\frac{\Delta^{r-1}u_1}{\Delta^{r-1}u_0} > 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

for, since $\quad \dfrac{\Delta^r u_0}{\Delta^{r-1}u_0} = \dfrac{\Delta^{r-1}u_1 - \Delta^{r-1}u_0}{\Delta^{r-1}u_0} = \dfrac{\Delta^{r-1}u_1}{\Delta^{r-1}u_0} - 1$, and

$$-1 < \operatorname{Lim}\frac{\Delta^r u_0}{\Delta^{r-1}u_0} < 1,$$

the condition of convergency is satisfied. It is seen that Rice's two cases are contained here, the first in the first, and the second in the second condition. It was not the object here to go into the question of how far the series is convergent, but only to point out the possibility of its convergence. The fuller meaning of the second condition, and the possibility of farther conditions by finer tests might form a subject of future investigation.

§5. THE Δu SERIES EXPRESSED IN TERMS OF THE δv'S.

Expressing the series $u_0 + \binom{\nu^1}{1}\Delta u_0 + \binom{\nu^1}{2}\Delta^2 u_0 + \ldots\ldots \binom{\nu^1}{r}\Delta^r u_0 + \ldots\ldots$ in terms of the δv's by means of (7), the collected coefficient c_r of $\delta^r v_0$ becomes

$$c_r = \sum_{\kappa=0}^{\kappa=r-1} \binom{\varkappa+1}{\nu^1} \sum_{\mu=1}^{\mu=r-\kappa} \binom{\lambda}{\mu} \Delta^\kappa(0, r-\mu) \ldots\ldots\ldots\ldots \ldots(14).$$

By means of the relations

$$\binom{\rho+\sigma}{r} = \binom{\rho}{0}\binom{\sigma}{r} + \binom{\rho}{1}\binom{\sigma}{r-1} + \ldots\ldots \binom{\rho}{r-1}\binom{\sigma}{1} + \binom{\rho}{r}\binom{\sigma}{0} \ldots\ldots\ldots(15)$$

and $\Delta^\kappa(0, \sigma) = 0$, where $\sigma < \varkappa$, it can be reduced to

$$c_r = \sum_{\kappa=1}^{\kappa=r} \binom{\nu^1}{\varkappa} \Delta^\kappa(0, r) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16).$$

$\Delta^\kappa(0, r)$ is of the form $a_0\lambda^r + a_1\lambda^{r-1} + \ldots\ldots a_{r-\kappa}\lambda^\kappa$. $\binom{\nu^1}{\varkappa}\Delta^\kappa(0, r)$ is therefore an integral function of λ and ν, of degree $r-1$ in λ, and of degree $\varkappa$ in ν; and c_r is integral and of no higher degree than $r-1$ in λ, and of degree r in ν. We shall show that it does not contain λ at all. $\binom{\varkappa\lambda}{r}$ is an integral function of $\varkappa\lambda$ of degree r, and is divisible by $\varkappa\lambda$; it is of the form $(\varkappa\lambda)^r+a_1(\varkappa\lambda)^{r-1}+\dots\, a_{r-1}(\varkappa\lambda)$ $=\varkappa^r\lambda^r+\varkappa^{r-1}a_1\lambda^{r-1}+\dots\,\varkappa a_{r-1}\lambda$; $\binom{(\varkappa-1)\lambda}{r}$ is therefore of the form $(\varkappa-1)^r\lambda^r$ $+(\varkappa-1)^{r-1}a_1\lambda^{r-1}+\dots(\varkappa-1)a_{r-1}\lambda$. Hence the coefficient of λ^σ in $\Delta^\kappa\,(0,\,r)$ is

$$a_{r-\sigma}\frac{\varkappa^\sigma-\varkappa(\varkappa-1)^\sigma+\frac{\varkappa(\varkappa-1)}{2\,!}(\varkappa-2)^\sigma-\dots(-1)^{\kappa-1}\varkappa.1^\sigma}{r\,!}$$

$$=\frac{1}{r(r-1)\dots(\sigma+1)}a_{r-\sigma}A_{(\kappa)+\sigma-\kappa}$$

where $A_{(\kappa)+\sigma-\kappa}$ is the coefficient of $x^{\sigma-\kappa}$ in $(e^x-1)^\kappa$, and where this is zero if $\sigma<\varkappa$,* which proves the statement that $\Delta^\kappa\,(0,\,r)$ contains the factor λ^κ.

§6. THE COEFFICIENT c_r OF $\delta^r v_0$.

We have shown that c_r is of the form,

$$c_r=f_r(\nu,\,r)+f_{r,}\,(\nu,\,r)\lambda+\dots f_1(\nu,\,r)\lambda^{r-1}\dots\dots\dots\dots(17),$$

the indices denoting the degree in ν; we shall prove that

$$f_\kappa\,(\nu,\,r)\equiv0,\ \varkappa=1,\,2,\,\dots r-1,\text{ and }c_r\equiv f_r(\nu,\,r)\equiv\binom{\nu}{r}\ \dots\dots(18).$$

The equation $c_r-\binom{\nu}{r}=0$, or

$$f_1(\nu,\,r)\lambda^{r-1}+f_2(\nu,\,r)\lambda^{r-2}+\dots f_r(\nu,\,r)-\binom{\nu}{r}=0\ \dots\dots\dots(19),$$

is satisfied by r values of λ,

$$\nu=\varkappa\lambda,\ \lambda=\frac{\nu}{\varkappa},\ \nu^1=\varkappa,\ \varkappa=1,\,2,\,\dots r\dots\dots\dots\dots(20).$$

We have when $\nu^1=\varkappa$,

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*See "Note on Integral and Integro-Geometric Series," l. c.

$$c_r=\binom{\varkappa}{1}\Delta(0,\,r)+\binom{\varkappa}{2}\Delta^2(0,\,r)=\;\ldots\ldots\binom{\varkappa}{\varkappa}\Delta^{\kappa}\,(0,\,r)\;\ldots\ldots\ldots(21).$$

The terms beyond that containing $\binom{\nu^1}{\varkappa}=\binom{\varkappa}{\varkappa}$ in c_r all vanish because they contain the factor $(\nu-\lambda\varkappa)=0$. The coefficient of $\binom{\frac{\rho\nu}{\kappa}}{r}$ in c_r is, $(\rho<\varkappa)$,

$$\binom{\varkappa}{\rho}-\binom{\varkappa}{\rho+1}\binom{\rho+1}{1}+\binom{\varkappa}{\rho+2}\binom{\rho+2}{2}-\ldots\ldots(-1)^{\sigma}\binom{\varkappa}{\rho+\sigma}\binom{\rho+\sigma}{\sigma}$$
$$+\;\ldots\ldots\;(-1)^{\kappa-\rho}\binom{\varkappa}{\varkappa}\binom{\varkappa}{\varkappa-\rho}\ldots\ldots\ldots\ldots\ldots\ldots(22).$$

The general term of this is $(-1)^{\sigma}\binom{\varkappa}{\rho+\sigma}\binom{\rho+\sigma}{\sigma}=(-1)^{\sigma}\binom{\varkappa}{\rho}\binom{\varkappa-\rho}{\sigma}$ and (22) becomes

$$\binom{\varkappa}{\rho}\sum_{\sigma=0}^{\sigma=\kappa-\rho}(-1)^{\sigma}\binom{\varkappa-\rho}{\sigma}=\binom{\varkappa}{\rho}(1-1)^{\kappa-\rho}=0\;\ldots\ldots\ldots\ldots(23),$$

and this is true for all values of ρ, from $\rho=1$, to $\rho=\varkappa-1$; only the single value $\binom{\varkappa}{\varkappa}\binom{\nu}{r}=\binom{\nu}{r}$ of c_r is left, and

$$c_r=\left[\begin{matrix}\nu\\ r\end{matrix}\right],\ \text{for}\ \lambda=\frac{\nu}{\varkappa},\ \varkappa=1,\,2,\,\ldots\ldots r\ldots\ldots\ldots\ldots\ldots(24),$$

hence (19) is an identity, and

$$c_r\equiv\left[\begin{matrix}\nu\\ r\end{matrix}\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25).$$

§7. The Sum of the Series.

By means of the result of §6, we can express the sum of the series (11), viz:

$$u_0+\left[\begin{matrix}\nu^1\\ 1\end{matrix}\right]\Delta u_0+\left[\begin{matrix}\nu^1\\ 2\end{matrix}\right]\Delta^2u_0+\ldots\ldots\left[\begin{matrix}\nu^1\\ r\end{matrix}\right]\Delta^ru_0+\ldots\ldots=v_0+\left[\begin{matrix}\nu\\ 1\end{matrix}\right]\delta v_0$$
$$+\ldots\ldots+\ldots\ldots\left[\begin{matrix}\nu^1\\ r\end{matrix}\right]\delta^rv_0+\ldots\ldots\left[\begin{matrix}\nu\\ \nu\end{matrix}\right]\delta^{\nu}\,v_0=v_{\nu}=f(x+\frac{\nu}{\lambda}),\ \text{or}$$

$$f(x+\frac{\nu}{\lambda})=f(x)+\frac{\nu}{\lambda}\Delta f(x)+\frac{\frac{\nu}{\lambda}\left[\frac{\nu}{\lambda}-1\right]}{2\,!}\Delta^2f(x)$$
$$+\ldots\ldots\frac{\frac{\nu}{\lambda}\left[\frac{\nu}{\lambda}-1\right]\ldots\ldots\left[\frac{\nu}{\lambda}-r+1\right]}{r\,!}\Delta^rf(x)+\ldots\ldots\ (26).$$

If we use the symbolic abbreviation before suggested with the understanding that by it we mean only the right member of (26), then we have proved,

$$f(x+\frac{\nu}{\lambda})=(1+\Delta)^{\nu/\lambda}f(x)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27).$$

§8. THE FORMULA (27) PROVED BY SYMBOLIC METHODS.

Immediately after the writer had found the previous non-symbolic proof that the formula $f(x+y)=(1+\Delta)^y f(x)$ holds for a commensurable fractional y, he discovered by symbolic methods a form of proof which is the opposite extreme of simplicity as compared with the non-symbolic. The two proofs confirm each other. We have:

$$\Delta u_0=((1+\delta)^\lambda-1)v_0.$$

$$\Delta u_1=((1+\delta)^\lambda-1)(1+\delta)^\lambda v_0.$$

$$\Delta u_2=((1+\delta)^\lambda-1)(1+\delta)^{2\lambda} v_0.$$

. . . .

$$\Delta u_r=((1+\delta)^\lambda-1)(1+\delta)^{r\lambda} v_0.$$

. . . .

$$\Delta^2 u_0=((1+\delta)^\lambda-1)^2 v_0.$$

$$\Delta^3 u_0=((1+\delta)^\lambda-1)^3 v_0.$$

. . . .

$$\Delta^r u_0=((1+\delta)^\lambda-1)^r v_0. \tag{28}$$

Hence by the use of the formulas, (28), and confining the meaning of the symbol $(1+\Delta)^{\nu/\lambda}$ as before,

$$(1+\Delta)^{\nu/r}u_0=u_0+\binom{\nu^1}{1}\Delta u_0+\ldots\binom{\nu^1}{r}\Delta^r u_0+\ldots$$

$$=v_0+\binom{\nu^1}{1}((1+\delta)^\lambda-1)v_0+\ldots\binom{\nu^1}{r}((1+\delta)^\lambda-1)^r v_0+\ldots$$

$$=(1+(1+\delta)^\lambda-1)^{\nu^1}v_0=(1+\delta)^{\nu^1\lambda}v_0=(1+\delta)^\nu v_0=v_\nu=f(x+\frac{\nu}{\lambda}),$$

or $f(x+\frac{\nu}{\lambda})=(1+\Delta)^{\nu/\lambda}f(x)$.

§9. EXTENSION TO NEGATIVE EXPONENTS.

Confining the symbols as before, and the series to convergency, and n and r to positive integers, $n>r$,

$$f(x+(n-r))=(1+\Delta)^{n-r}f(x)=f((x+n)-r)\ldots\ldots\ldots\ldots\ldots(29).$$

Also $(1+\Delta)^{-r}f(x+n)=f(x+n)-r\Delta f(x+n)+\ldots$

$$=(1+\Delta)^n(f(x)-r\Delta f(x)+\ldots)$$

$$=(1+\Delta)^n(1+\Delta)^{-r}f(x)$$

$$=(1+\Delta)^{n-r}f(x) \tag{30}$$

Comparing (29) and (30) we have

$$f((x+n)-r)=(1+\Delta)^{-r}f(x+n) \ldots\ldots\ldots\ldots\ldots (31),$$

or denoting $(x+n)$ by a single letter,

$$f(x-r)=(1+\Delta)^{-r}f(x) \ldots\ldots\ldots\ldots\ldots (32).$$

In order to extend to a negative fractional commensurable index, we choose, $-\nu=\varkappa\lambda$, $\varkappa=1, 2, \ldots . r$, in §6, and come out with $c_r=\binom{-\nu}{r}$ always, where we are supposed to have found the value of the simplest series $(1+\Delta)^{-\nu/\lambda}f(x)$ in terms of δv's, and thus we have

$$(1+\Delta)^{-\nu/\lambda}f(x)=(1+\delta)^{-\nu}f(x) \ldots\ldots\ldots\ldots\ldots (33).$$

Or as in §8, by symbolic operations we arrive at

$$(1+\Delta)^{-\nu^1}u_0=(1+\delta)^{-\nu^1\lambda}v_0=(1+\delta)^{-\nu}v_0.$$

By reasoning analogous to that by which (32) was found,

$$(1+\delta)^{-\nu}f(x)=f(x-\frac{\nu}{\lambda}) \ldots\ldots\ldots\ldots\ldots (34).$$

Comparing (33) and (34),

$$f(x-\frac{\nu}{\lambda})=(1+\Delta)^{-\nu/\lambda}f(x) \ldots\ldots\ldots\ldots\ldots (35).$$

§10. EXTENSION TO AN INCOMMENSURABLE y.

Here we have to use limits. Let y be incommensurable, y^1 commensurable, and let $y^1=y$. Then by a combination of the previous paragraphs,

$$f(x+y^1)=(1+\Delta)^{y^1}f(x), \text{ and}$$

$$\mathop{\mathrm{Lim}}\limits_{y^1=y} f(x+y)^1)=\mathop{\mathrm{Lim}}\limits_{y^1=y}(1+\Delta)^{y^1}f(x), \text{ or}$$

$$f(x+y)=(1+\Delta)^{y}f(x) \ldots\ldots\ldots\ldots\ldots (36).$$

Hence within the limits of convergence, and for the confined meaning of the symbol $(1+\Delta)^y$, we have proved formula (36) to hold for all real values whatsoever of y.

Syracuse University, 1 December, 1900.